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NUMERICAL SIMULATION OF STOCHASTIC PROCESSES

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FOREWORD

Stochastic processes are utilized to represent many diverse phenomena. This report describes a stochastic process generator useful for simulation studies with the help of the computer.

This report has been reviewed by John H. Walker, Jr.

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ABSTRACT

A method has been derived to simulate a one-dimensional stationary stochastic process with a given autocorrelation function by a finite trigonometric sum. The coefficients of the latter are uncorrelated random numbers. A rigorous estimate of the degree of approximation to the autocorrelation function is given. The method is quite general and does not require the power spectrum to be rational.

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1. INTRODUCTION

Stochastic processes (see Refs. 1, 2) are used to represent phenomena in many diverse fields.

It is of importance to possess numerical algorithms that generate stochastic processes in order to conduct simulation studies on a computer.

In this report we adhere to the convention of underlining all random quantities.

The method to be outlined uses a sequence of independent random variables to generate a stationary random process with a specified autocorrelation. Thus we may simulate the process on a computer with the aid of a "random number generator". This report confines itself to discussing one-dimensional stochastic processes.

2. Simulation of a one-dimensional weakly stationary stochastic process with given autocorrelation.

Let the mean of a stationary process $\underline{x}(t)$ be zero and the autocorrelation be $R_x(\tau)$. The power spectrum (see ref. 1, p. 338) is given by

$$S_x(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} R_x(\tau) d\tau \tag{1}$$

The inverse relationship is

$$R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} \; S_x(\omega)d\omega \tag{2}$$

We have here and forthwith assumed that there is no line spectrum.

We may now construct a stochastic process $\hat{\underline{x}}(t)$ which has the given mean, possesses approximately the given autocorrelation function and depends only on a discrete number of random variables. We proceed by dividing the frequency range into discrete intervals. Let us denote the size of the interval by ω_o. This increment is to be small in some sense to be specified later. Define the quantities

$$\alpha_n = \frac{1}{2\pi} \int_{n\omega_o - \frac{1}{2}\omega_o}^{n\omega_o + \frac{1}{2}\omega_o} S_x(\omega)d\omega \tag{3}$$

$$n = 0, \pm 1, \ldots$$

These quantities are obviously non-negative and represent the power in the frequency range of $n\omega_o - \frac{1}{2}\omega_o$ to $n\omega_o + \frac{1}{2}\omega_o$. Now take a discrete set of complex random variables which satisfy

$$\underline{c}_n = \underline{c}^*_{-n} \tag{4}$$

$$E\{\underline{c}_o\} = 0 \tag{5}$$

$$E\{\underline{c}_n\} = 0, \; n = \pm 1, \ldots \tag{6}$$

$$E\{\underline{c}_n \underline{c}^*_m\} = 0, \; n \neq m \tag{7}$$

$$E\{|\underline{c}_n|^2\} = \alpha_n, \; n = 0, \pm 1, \ldots \tag{8}$$

Now, of course, these conditions do not determine the random variables $\underline{c}_n$ uniquely. However all the stochastic processes defined by

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} \underline{c}_n e^{in\omega_0 t} \tag{9}$$

possess certain common properties. We have

$$E\{\hat{\underline{x}}(t)\} = 0 = E\{\underline{x}(t)\} \tag{10}$$

$$E\{\hat{\underline{x}}(t)^2\} = \sum_{n=-\infty}^{\infty} E\{|\underline{c}_n|^2\} = \sum_{n=-\infty}^{\infty} \alpha_n$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)d\omega = E\{\underline{x}(t)^2\} \tag{11}$$

Thus the processes $\hat{\underline{x}}(t)$ have the same mean and total power as the process $\underline{x}(t)$.

Now let us investigate the autocorrelation of $\hat{\underline{x}}(t)$. We have from Eqs. 7, 8

$$R_{\hat{x}}(\tau) = E\{\hat{\underline{x}}(t+\tau)\hat{\underline{x}}(t)\}$$

$$= \sum_{n=-\infty}^{\infty} E\{|\underline{c}_n|^2\} e^{in\omega_0\tau} = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega_0\tau} \tag{12}$$

Thus the $\hat{\underline{x}}(t)$ are stationary in the weak sense, but their autocorrelation differs from $R_x(\tau)$. Let us estimate the difference. How well does $R_{\hat{x}}(\tau)$ approximate $R_x(\tau)$? We may write

$$R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) e^{i\omega\tau} \, d\omega$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{n\omega_o - \frac{1}{2}\omega_o}^{n\omega_o + \frac{1}{2}\omega_o} S_x(\omega) e^{i\omega\tau} \, d\omega \tag{13}$$

Each term in Eq. 13 may be written

$$\frac{1}{2\pi} \int_{n\omega_o - \frac{1}{2}\omega_o}^{n\omega_o + \frac{1}{2}\omega_o} S_x(\omega) e^{i\omega\tau} \, d\omega = \alpha_n e^{in\omega_o\tau}$$

$$+ \frac{i}{\pi} \int_{n\omega_o - \frac{1}{2}\omega_o}^{n\omega_o + \frac{1}{2}\omega_o} S_x(\omega) e^{\frac{1}{2} i\tau(\omega + n\omega_o)} \sin\left[\frac{(\omega - n\omega_o)\tau}{2}\right] d\omega \tag{14}$$

Therefore we may write for the difference of the desired auto-correlation and the simulated one

$R_x(\tau) - R_x^{\wedge}(\tau) =$

$$= \frac{i}{\pi} \sum_{n=-\infty}^{\infty} \int_{n\omega_o - \frac{1}{2}\omega_o}^{n\omega_o + \frac{1}{2}\omega_o} S_x(\omega) e^{\frac{1}{2} i\tau(\omega + n\omega_o)} \sin\left[\frac{(\omega - n\omega_o)\tau}{2}\right] d\omega$$

$$= -\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \int_{n\omega_o - \frac{1}{2}\omega_o}^{n\omega_o + \frac{1}{2}\omega_o} S_x(\omega) \sin\left[\frac{(\omega + n\omega_o)\tau}{2}\right] \sin\left[\frac{(\omega - n\omega_o)\tau}{2}\right] d\omega \tag{15}$$

The magnitude of the final expression may be estimated without difficulty. When

$$n\omega_o - \frac{1}{2}\omega_o \leqslant \omega \leqslant n\omega_o + \frac{1}{2}\omega_o \tag{16}$$

we have the inequality

$$\left| \sin \frac{(\omega - n\omega_o)\tau}{2} \right| \leqslant \frac{1}{4}\omega_o |\tau| \tag{17}$$

Consequently by the first mean value theorem of the integral calculus, we have the following inequality

$$\left| \int_{n\omega_o - \frac{1}{2}\omega_o}^{n\omega_o + \frac{1}{2}\omega_o} S_x(\omega) \sin\left[\frac{(\omega + n\omega_o)\tau}{2}\right] \sin\left[\frac{(\omega - n\omega_o)\tau}{2}\right] d\omega \right|$$

$$\leqslant \frac{1}{4}\,\omega_o|\tau| \int_{n\omega_o - \frac{1}{2}\omega_o}^{n\omega_o + \frac{1}{2}\omega_o} S_x(\omega)\,d\omega \tag{18}$$

Equation 15 thus yields the inequality,

$$\left| R_x(\tau) - R_{\hat{x}}(\tau) \right| \leqslant \frac{1}{2}\,\omega_o|\tau| \; \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)\,d\omega$$

$$= \frac{1}{2}\,\omega_o|\tau|\,R_x(0)$$

$$= \frac{1}{2}\,\omega_o|\tau| \times \text{Total Power of } \underline{x}(t) \tag{19}$$

We see from this inequality that in order to approximate $R_x(\tau)$ over a range $|\tau| \leqslant T$, we must choose a frequency increment ω_o, such that

$$\omega_o \ll \frac{2}{T} \tag{20}$$

Furthermore we see that however small we choose ω_o, $R_{\hat{x}}(\tau)$ will not approximate $R_x(\tau)$ for all τ. This is connected with the fact the process $\hat{\underline{x}}(t)$ involves only a discrete set of frequencies.

The inequality 19 gives us the effect of the frequency discretization on the autocorrelation function. For the purpose of numerical simulation we must also estimate the effect of truncating the series in Eq. 9. In contrast to the frequency discretization, the truncation affects the total power according to Eq. 11. If we truncate after $|n| = N-1$, the power will be reduced according to Eq. 11 by

$$\sum_{|n| \geq N} \alpha_n = \frac{1}{\pi} \int_{N\omega_o - \frac{1}{2}\omega_o}^{\infty} S_x(\omega)\,d\omega \tag{21}$$

3. <u>An example</u>

Suppose we wish to simulate a stochastic process with autocorrelation function

$$R_x(\tau) = \sigma^2 e^{-\beta|\tau|} \tag{22}$$

Its power spectrum is (see Ref. 1, p. 340)

$$S_x(\omega) = \frac{2\beta\sigma^2}{\beta^2 + \omega^2} \tag{23}$$

The total power is

$$R_x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)\,d\omega = \sigma^2 \tag{24}$$

We may define a correlation distance ℓ by

$$\ell = \frac{1}{\beta} \tag{25}$$

Suppose we wish to represent the autocorrelation function to within 100ϵ % of the total power over a distance of m correlation distances. The maximum lag is given by

$$|\tau|_{max} = \frac{m}{\beta} \tag{26}$$

The error in the autocorrelation function due to discretization satisfies according to Eq. 19

$$e_D \leqslant \frac{m}{2} \frac{\omega_o}{\beta} \sigma^2 \tag{27}$$

The error due to truncation satisfies according to Eq. 21

$$e_T \leqslant \frac{2\sigma^2}{\pi} \int_{N\omega_o - \frac{1}{2}\omega_o}^{\infty} \frac{\beta\, d\omega}{\beta^2 + \omega^2}$$

$$= \frac{2\sigma^2}{\pi} \cdot \int_{\frac{1}{\beta}(N\omega_o - \frac{1}{2}\omega_o)}^{\infty} \frac{d\nu}{1 + \nu^2} \tag{28}$$

Now

$$\int_{\frac{1}{\beta}(N\omega_o - \frac{1}{2}\omega_o)}^{\infty} \frac{d\nu}{1 + \nu^2} < \frac{\beta}{(N-1)\,\omega_o} \tag{29}$$

and therefore

$$e_T < \frac{2}{\pi} \cdot \frac{\beta}{\omega_o} \cdot \frac{1}{N-1} \cdot \sigma^2 \tag{30}$$

We require that

$$\frac{m}{2} \cdot \frac{\omega_o}{\beta} + \frac{2}{\pi} \cdot \frac{\beta}{\omega_o} \cdot \frac{1}{N-1} = \epsilon \tag{31}$$

We would like to satisfy this equality with as small a value of N as possible. Minimizing the expression on the left with respect to $\frac{\omega_o}{\beta}$ yields

$$\frac{\omega_o}{\beta} = \frac{2}{\sqrt{\pi m(N-1)}} \tag{32}$$

Substituting this value in Eq. 31 yields

$$2\sqrt{\frac{m}{\pi}} \quad \frac{1}{\sqrt{N-1}} = \epsilon \tag{33}$$

or

$$N-1 = \frac{4}{\pi} \cdot \frac{m}{\epsilon^2} \tag{34}$$

or

$$N \approx 1.27 \frac{m}{\epsilon^2} \tag{35}$$

If, for example, we wish a precision of 10% over 100 correlation distances, the number of terms required is

$$N \approx 1.27 \quad \frac{100}{.01} = 12700 \tag{36}$$

4. Does the process $\hat{\underline{x}}(t)$ approximate the process $\underline{x}(t)$?

We might be tempted to believe that a process $\hat{\underline{x}}(t)$ constructed in the aforementioned way, approximates in some sense the process $\underline{x}(t)$. However that is only true if

$$\underline{c}_n = \int_{-\infty}^{\infty} \underline{x}(t) \frac{\sin(\frac{1}{2}\omega_o t)}{\pi t} e^{-in\omega_o t} dt \tag{37}$$

(see Ref. 1, p. 461). These $\underline{c}_n$ satisfy all the conditions expressed by Eqs. 4-8 and the process $\hat{\underline{x}}(t)$ then approximates $\underline{x}(t)$ in a mean square sense as discussed in the reference.

5. Strict sense stationarity and normality

It may be useful for some applications to be able to generate stochastic processes that are stationary in the strong sense. The stochastic processes $\hat{\underline{x}}(t)$ as defined by Eqs. 3-9 are in general only stationary in the weak sense. We may show this by regarding the following example: Write

$$\underline{c}_n = \underline{\rho}_n e^{i\underline{\theta}_n} \tag{38}$$

where

$$\underline{\rho}_n = \underline{\rho}_{-n} \tag{39}$$

$$\underline{\theta}_n = -\underline{\theta}_{-n} \tag{40}$$

Let the $\underline{\rho}_n$ and $\underline{\theta}_n$ be mutually independent random variables. Let the $\underline{\theta}_n$ have the probability density function

$$f_\theta(\theta) = \frac{1}{2\pi}(1 + \cos 3\theta)$$
$$-\pi \leqslant \theta < \pi \tag{41}$$

Let the $\underline{\rho}_n$ satisfy

$$E\{\underline{\rho}_o\} = 0 \tag{42}$$

$$E\{\underline{\rho}_n\} = 0 \qquad n = 1, \ldots \tag{43}$$

$$E\{\underline{\rho}_n^2\} = \alpha_n \qquad n = 0, 1, \ldots \tag{44}$$

and let

$$E\{\underline{\rho}_n^3\} = \beta_n \tag{45}$$

where the β_n are any sequence of numbers not all zero. It follows that the Eqs. 4-8 are satisfied and the process

$$\hat{\underline{x}}(t) = \sum_{n=-\infty}^{\infty} \underline{c}_n e^{in\omega_o t} \tag{46}$$

is weakly stationary. It is however not stationary in the strong sense as may be shown by observing that

$$E\left\{\hat{\underline{x}}(t)^3\right\} = E\left\{\left(\sum_{n=-\infty}^{\infty} \underline{\rho}_n e^{i\underline{\theta}_n} e^{in\omega_o t}\right)^3\right\}$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} \beta_n e^{3in\omega_o t} \tag{47}$$

which is not constant.

It may be shown that if the $\underline{\rho}_n$ and $\underline{\theta}_n$ are mutually independent, the $\underline{\rho}_n$ satisfying Eqs. 39, 42-44 and the $\underline{\theta}_n$ are

uniformly distributed in $-\pi$ to π and satisfying Eq. 40 the process $\hat{\underline{x}}(t)$ of Eq. 46 is stationary in the strict sense.

It is possible to generate a normal process by the method described (for the existance of normal processes see Ref. 2, p. 72). In this case weak stationarity implies strong stationarity. Normality is often assumed if the process to be simulated is imperfectly known. Various theoretical results concerning sample distributions may be derived for normal process.

To generate a normal process let us write

$$\underline{c}_n = \frac{1}{2}(\underline{a}_n - i\underline{b}_n) \tag{48}$$

$$\underline{a}_n = \underline{a}_{-n}, \quad \underline{b}_n = -\underline{b}_{-n} \tag{49}$$

We may rewrite Eq. 9 as

$$\hat{\underline{x}}(t) = \frac{1}{2}\underline{a}_o + \sum_{n=1}^{\infty}(\underline{a}_n \cos n\omega_o t + \underline{b}_n \sin n\omega_o t) \tag{50}$$

Let us choose the $\underline{a}_n$ and $\underline{b}_n$ as mutually independent normal random variables satisfying

$$E\{\underline{a}_o\} = 0 \tag{51}$$

$$E\{\underline{a}_n\} = E\{\underline{b}_n\} = 0, \quad n = 1, \ldots \tag{52}$$

$$E\{\underline{a}_n^2\} = 4\alpha_n, \quad n = 0, 1, \ldots \tag{53}$$

$$E\{\underline{b}_n^2\} = 4\alpha_n, \quad n = 1, \ldots \tag{54}$$

These equations imply Eqs. 4-8. Now since $\hat{\underline{x}}(t)$ is a linear combination of normal random variables, it is a normal process and stationary in the strict sense.

We may prove that the normal process we have just described has the property of "random phase", namely the random variables $\underline{\theta}_n$ defined by

$$\operatorname{tg} \underline{\theta}_n = \frac{\underline{b}_n}{\underline{a}_n} \tag{55}$$

$$-\pi \leqslant \theta_n < \pi \tag{56}$$

are mutually independent and uniformly distributed. Indeed the random variable

$$\underline{z} = \frac{\underline{b}_n}{\underline{a}_n} \tag{57}$$

is distributed according to Student's distribution of one degree of freedom (Ref. 3, p. 237).

Student's distribution reduces for one degree of freedom to Cauchy's distribution (Ref. 3, p. 246). The probability density function is

$$f_z(z) = \frac{1}{\pi} \frac{1}{1 + z^2} \tag{58}$$

From this we easily derive that the probability density function for $\underline{\theta}_n$ is

$$f_{\theta}(\theta) = \begin{cases} \frac{1}{\pi}, & -\pi \leqslant \theta < \pi \\ 0, & \text{otherwise} \end{cases} \tag{59}$$

Random phases are often made a requirement in representing various phenomena in the time domain.

6. CONCLUSION

We have presented a method to simulate stationary stochastic processes numerically. The method is quite general and does not depend on the power spectrum being rational. It only requires a random number generator and adequate computing power.

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13. ABSTRACT

A method has been derived to simulate a one-dimensional stationary stochastic process with a given autocorrelation function by a finite trigonometric sum. The coefficients of the latter are uncorrelated random numbers. A rigorous estimate of the degree of approximation to the autocorrelation function is given. The method is quite general and does not require the power spectrum to be rational.